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# Convergence of Numerical Solutions of the Data Assimilation Problem for the Atmospheric General Circulation Model

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## Abstract

*We consider a two-layer quasigeostrophic model of the general atmospheric circulation. It is assumed that there are field measurements of air velocity. These observations are used to find the unknown initial state of the model. The discrepancy between the observed values and the model results is measured by a cost function value. We prove the solvability of the optimization problem for positive values of the regularization parameter. The system of equations is approximated by an explicit spectral-difference scheme. A theorem is proved that the numerical solutions of the data assimilation problem converge to its exact solutions.*

**Keywords:** Optimization problem, model of the atmospheric dynamics, spectral-difference scheme.

## Introduction

A rigorous mathematical analysis and justification of the variational data assimilation procedure includes the study of such issues as the existence of solutions to the optimization problem and the convergence of numerical solutions to exact solutions. In this paper we study these issues in relation to the two-layer baroclinic quasi-geostrophic atmospheric general circulation model. The main variables of the model are barotropic and baroclinic components of the stream function, but the stream function is not one of variables for which in meteorology are carried out the field observations. For this reason, it is assumed that the measurements of the velocity of air are known. The initial state is chosen as the model parameter to be determined because the initialization problem is one of the best known and most commonly solved in practice.

Note that the convergence of numerical solutions of the data assimilation problem earlier has been studied for the quasi-geostrophic models in a rectangular region under the assumptions that the equations are approximated by the implicit[1] or semi-explicit[2] finite-difference schemes and the observations on the ocean surface elevation are given .

## Material and Methods

**The atmospheric general circulation model:** Let $S$ be a two-dimensional sphere of radius $R$, $\theta \in [0,2\pi)$ be the longitude, $\varphi \in [-\pi/2;\pi/2]$ be the latitude, $\Omega$ be the the angular velocity of the Earth rotation. By $l = 2\Omega \sin\varphi$ we denote the Coriolis parameter, $\Delta = \frac{1}{R^2\cos^2\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{R^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right)$ is the Laplace-Beltrami operator, $J(u,v) = \frac{1}{R^2\cos\varphi}\left(\frac{\partial u}{\partial\theta}\frac{\partial v}{\partial\varphi} - \frac{\partial v}{\partial\theta}\frac{\partial u}{\partial\varphi}\right)$ is the Jacobian.

The atmosphere is divided vertical into two layers, the first layer correspond to the pressure from 0 to 500 mb and the second layer correspond to the pressure from 500 to 1000 mb, $\psi_1 = \psi_1(\theta,\varphi,t)$, $\psi_2 = \psi_2(\theta,\varphi,t)$ are the stream functions within the first and the second layers, $x_1 = (\psi_1 + \psi_2)/2$, $x_2 = (\psi_2 - \psi_1)/2$ are barotropic and baroclinic components of the stream function, $x = (x_1, x_2)$. We consider the atmospheric general circulation model [3]:

$$\frac{\partial \Delta x_1}{\partial t} + J(x_1, \Delta x_1 + l) + J(x_2, \Delta x_2) = \mu\Delta^2 x_1 - \sigma\Delta(x_1 + x_2) + f_1, \quad (1)$$

$$\frac{\partial(\Delta - \alpha)x_2}{\partial t} + J(x_2, \Delta x_1 + l) + J(x_1, \Delta x_2) = \mu\,\Delta^2 x_2 - \sigma\Delta(x_1 + x_2) + \alpha J(x_1, x_2) - \mu_1\Delta x_2 + \sigma_1 x_2 + f_2, \quad (2)$$

$$x\big|_{t=0} = x_0. \quad (3)$$

Here $\sigma, \mu, \sigma_1, \mu_1, \alpha$ are positive constants and $f_1, f_2$ are given functions.

Introduce the real Hilbert space $L_2^0 = \left\{u(\theta,\varphi) \in L_2(S), \int_S u\,dS = 0\right\}$ with the scalar product $(u,v) = \int_S uv\,dS$ and the norm $\| u \| = (u,u)^{1/2}$. Associate the operator $(-\Delta)$ with the scale of Hilbert spaces $H^p = H^p(S)$, $p \in \mathbb{R}$, by assuming $H^p = \left\{u(\theta,\varphi) \in L_2^0, \| u \|_p = \| (-\Delta)^{p/2} u \| < +\infty\right\}$. For vector functions $x = (x_1, x_2)$ we introduce the spaces $V_p = V_p(S) = H^p \times H^p$ with the norm $\| x\|_p = \left(\| x_1\|_p^2 + \| x_2\|_p^2\right)^{1/2}$, where $V_0 = L_2^0 \times L_2^0$, $\| x\|_0 \equiv \| x\|$.

Let $0 < T < +\infty$ and $G = S\times(0,T)$. By $(\cdot,\cdot)_G$ and $\|\cdot\|_G$ we denote the scalar product and the norm in the space $L_2(G)$ respectively. Introduce the real spaces of the functions determined in $G$:

$$X = L_2\left(0,T;V_3\right),\quad Y = L_2\left(0,T;V_2\right),\quad Y_1 = L_2\left(0,T;V_1\right),$$

$$Z = L_2\left(0,T;V_{-1}\right),\quad W = \left\{x\in X, \frac{\partial x}{\partial t}\in Y_1\right\}.$$

We also introduce the bilinear forms $a_1(u,v)$, $a_2(u,v)$, $b_1(u,v)$ and the trilinear form $b_2(u,v,w)$ by setting

$$a_1(u,v) = \int_S \left(\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + \alpha u_2 v_2\right) dS,$$

$$a_2(u,v) = \int_S \{\mu(\Delta u_1 \Delta v_1 + \Delta u_2 \Delta v_2) + \mu_1 \nabla u_2 \nabla v_2 +$$
$$+\sigma\nabla(u_1+u_2)\nabla(v_1+v_2) + \sigma_1 u_2 v_2\} dS,$$

$$b_1(u,v) = \int_S \left[J(u_1,l)v_1 + J(u_2,l)v_2\right] dS,$$

$$b_2(u,v,w) = \int_S \{J(w_1,v_1)\Delta u_1 + J(w_2,v_2)\Delta u_1 + J(w_1,v_2)\Delta u_2 +$$
$$+J(w_2,v_1)\Delta u_2 + \alpha J(w_2,v_2)u_1\} dS.$$

The system (1)-(2) can be written as

$$a_1\left(\frac{\partial x}{\partial t}, y\right) + a_2(x,y) = b_1(x,y) + b_2(x,x,y) - (f,y)\quad \forall y\in V_2$$

where $f = (f_1, f_2)$.

Further the letter $c$ denotes various positive constants. We will need the following statements.

**Lemma 1.** *The inequalities hold* [3,4]

$$| b_2(u,v,w)| \le c\| u\|_2 \| v\|_2 \| w\|_2 \quad \forall u,v,w\in V_2,$$

$$\| \nabla u\|_{L_4(S)} \le 2^{1/4} \| u\|_1^{1/2} \| u\|_2^{1/2} \quad \forall u\in H^2,$$

$$|(J(w,u),\Delta v) + (J(w,v),\Delta u)| \le$$
$$\le c\| \nabla u\|_{L_4(S)} \| \nabla v\|_{L_4(S)} \| w\|_2 \quad \forall u,v,w\in H^2.$$

**Lemma 2:** *Let $B_0$, $B_1$, $B_2$ be three Banach spaces where $B_0 \subset B_1 \subset B_2$, $B_0$ and $B_2$ are reflexive, $B_0$ is compactly embedded into $B_1$, and $B_1$ is continuously embedded into $B_2$; let $W = \left\{x\in L_{p_0}(0,T;B_0),\ \frac{\partial x}{\partial t}\in L_{p_1}(0,T;B_2)\right\}$, where $T$ is finite and $1 < p_k < \infty$, $k = 0,1$. Then the embedding of $W$ into $L_{p_0}(0,T;B_1)$ is compact* [5].

**Theorem 1** [3]**:** *For all $x_0 \in V_2$ and $F\in Z$ the problem (1)-(3) has the unique solution $x\in W$, and the estimates hold*

$$\max_{0\le t\le T} \| x(t)\|_1 + \| x\|_Y \le c\left(\| x_0\|_1 + \| f\|_Z\right),$$

$$\max_{0\le t\le T} \| x(t)\|_2 + \| x\|_X \le c\left(\| x_0\|_2 + \| f\|_Z\right),\qquad \left\|\partial x/\partial t\right\|_{Y_1} \le c_1,$$

*where $c_1$ is a constant depending on $\| x_0\|_2$ and $\| f\|_Z$.*

Write the problem (1)-(3) down briefly as $\Phi(x) = \left(f;x_0\right)$. It follows from Theorem 1 that there exists a bounded inverse operator $\Phi^{-1}: Z\times V_2 \to W$ defined on all $Z\times V_2$.

**Lemma 3:** *Let $f\in Z$, $x_n = \Phi^{-1}\left(f;y_n\right)$, $x = \Phi^{-1}\left(f;x_0\right)$ and $y_n \to x_0$ weakly in $V_2$. Then $x_n \to x$ weakly in $W$.*

**Proof.** The sequence $\{y_n\}$ is bounded in $V_2$. On Theorem 1 the sequence $\{x_n\}$ is bounded in $W$. Choose a convergent subsequence in it: $x_n \to z$ weakly in $W$. By Lemma 2, then $x_n \to z$ strongly in $Y$. The space $C\left(\left[0,T\right];V_2\right)$ is continuously embedded [6] into $W$, so $y_n \to z(\theta,\varphi,0)$ weakly in $V_2$. Using the estimates of Lemma 1, we see that $b_2\left(x_n,x_n,y\right) \to b_2\left(z,z,y\right)$ weakly in $L_2\left(0,T\right)$ for all $y\in V_2$. Taking the limit as $n\to\infty$, we find that $z$ is a solution of (1)-(3), i.e. $z = x$.

**The data assimilation problem:** Let us assume that we know observation data for the velocity vector of air in the first and the second layers given by functions $u_k^0$, $v_k^0$, $k = 1,2$ on some measurable subset $G_0 \subset G$. Denote by $\chi$ the characteristic function of $G_0$ and extend $u_k^0$, $v_k^0$ onto $G\setminus G_0$ by zero. We associate to each solution of (1)-(3) the functions $\psi_1(x) = x_1 - x_2$, $\psi_2(x) = x_1 + x_2$, $u_k(x) = \frac{1}{R}\frac{\partial\psi_k(x)}{\partial\varphi}$,

$$v_k(x) = -\frac{1}{R\cos\varphi}\frac{\partial\psi_k(x)}{\partial\theta},\ k = 1,2.$$

Define the following cost functional on $W$:

$$I\left(x\right) = m_1\left\|\chi u_1(x) - u_1^0\right\|_G^2 + m_2\left\|\chi v_1(x) - v_1^0\right\|_G^2 +$$
$$+m_3\left\|\chi u_2(x) - u_2^0\right\|_G^2 + m_4\left\|\chi v_2(x) - v_2^0\right\|_G^2$$

where $m_1$, $m_2$, $m_3$, $m_4$ are non-negative weight coefficients.

Assume that the external forcing $f$ in the model is known and the observation data should be used for determination of the initial state $x_0$. Define on $V_2$ the functional

$$J_\lambda\left(x_0\right) = \lambda\left\|x_0 - x_0^a\right\|_2^2 + I\left(\Phi^{-1}\left(f;x_0\right)\right) \qquad (4)$$

where $\lambda \ge 0$ is a regularization parameter, $x_0^a \in V_2$ is a priori known approximate value of $x_0$.

Consider the following data assimilation problem: given an external action $f \in Z$, determine $x_0 \in V_2$ so that

$$J_\lambda(x_0) = \inf\{J_\lambda(y) \mid y \in V_2\}. \qquad (5)$$

Sufficient conditions for its solvability gives the theorem.

**Theorem 2:** *If $\lambda > 0$ and functions $u_k^0$, $v_k^0$, $k = 1,2$ belong to $L_2(G)$, then problem (5) has a solution.*

**Proof:** Denote $m = \inf\{J_\lambda(y) \mid y \in V_2\}$ and consider a sequence $\{y_n\}$ minimizing the functional $J_\lambda$, that is $\lim_{n\to\infty} J_\lambda(y_n) = m$. If $\lambda > 0$ then $\{y_n\}$ is bounded in $V_2$. Choose a subsequence $y_n \to x_0$ weakly convergent in $V_2$. Define $z_n = \Phi^{-1}(f; y_n)$ and $x = \Phi^{-1}(f; x_0)$. By Lemma 3 we have a convergence $z_n \to x$ weakly in $W$. It follows from Lemma 2 that $z_n \to x$ strongly in $Y$. Then $u_k(z_n) \to u_k(x)$ and $v_k(z_n) \to v_k(x)$, $k = 1,2$, strongly in $L_2(G)$. Thus, $\lim_{n\to\infty} I(z_n) = I(x)$. By the property of weak lower semicontinuity of norms we have $\|x_0 - x_0^a\|_2 \le \limsup_{n\to\infty} \|y_n - x_0^a\|_2$, therefore $J_\lambda(x_0) \le m$. Taking into account the definition of $m$, we conclude that $x_0$ is a solution of (5). Since $J_\lambda(x_0) = m$, then $\| y_n \|_2 \to \| x_0 \|_2$ so $y_n \to x_0$ strongly in $V_2$.

**The approximate data assimilation problem:** Now consider a discrete method for determination of approximate solutions to the problem (5). Let $\mathcal{H}_n$ be the eigensubspace of the Laplace-Beltrami operator corresponding to the eigenvalue $\Lambda_n = n(n+1)$ and spanned onto spherical harmonics $Y_{mn}(\theta,\varphi)$, $|m| \le n$. Denote $\mathcal{H}^N = \cup_{n=1}^N \mathcal{H}_n$, $\Xi^N = \mathcal{H}^N \times \mathcal{H}^N$ and also denote the operator of the orthogonal projection onto $\mathcal{H}^N$ by $P_N$. Let $\tau = T/K$ be the grid time step, $t_k = k\tau$, $k = \overline{0,K}$, $x^k$ is the approximate solution in the layer $t = t_k$. Further we assume that for varying $\tau$ and $N$ the inequality

$$\tau\Lambda_N = \tau N(N+1) \le (\mu - \nu)\mu^{-2} \qquad (6)$$

holds with some constant $\nu \in (0,\mu)$.

Approximate the problem (1)-(3) by the explicit spectral-difference scheme:

$$\begin{aligned}
&\Delta D_1^k/\tau + P_N J(x_1^k, \Delta x_1^k + l) + P_N J(x_2^k, \Delta x_2^k) \\
&\quad + \sigma\Delta(x_1^k + x_2^k) - \mu\Delta^2 x_1^k = q_1^k \in \mathcal{H}^N, \\
&(\Delta - \alpha) D_2^k/\tau + P_N J(x_2^k, \Delta x_1^k + l) + P_N J(x_1^k, \Delta x_2^k) \\
&-\alpha P_N J(x_1^k, x_2^k) + \sigma\Delta(x_1^k + x_2^k) - \mu\Delta^2 x_2^k + \mu_1 \Delta x_2^k - \sigma_1 x_2^k = q_2^k \in \mathcal{H}^N, \\
&x^k \in \Xi^N, \quad k = \overline{0,K}, \quad x^0 = \rho \in \Xi^N,
\end{aligned} \qquad (7)$$

where $D_j^k = x_j^{k+1} - x_j^k$, $j = 1,2$. Write down system (7) in a brief form $F(x) = (q;\rho)$, where the operator $F$ depends on $\tau$ and $N$, but for the sake of brevity, we omit this dependence. Equations (7) form a system linear with respect to $x^{k+1}$ with a nondegenerate matrix. Therefore, the operator $F$ is uniquely invertible on the whole $(\Xi^N)^K \times \Xi^N$. In order to extend the grid function $x = \{x^k\}_{k=0}^K$ onto the whole time segment $[0,T]$, we associate it with the function of a continuous argument $A(x)(\theta,\varphi,t) = \frac{t_{k+1} - t}{\tau} x^k(\theta,\varphi) + \frac{t - t_k}{\tau} x^{k+1}(\theta,\varphi)$ for $t \in [t_k, t_{k+1}]$.

We define on $\Xi^N$ the cost functional similar to the functional (4) by setting $S_\lambda(\rho) = \lambda\|\rho - x_0^a\|_2^2 + I\left(A\left(F^{-1}(q;\rho)\right)\right)$, where the external influence $q \in (\Xi^N)^K$ is considered to be known and fixed. Consider the following discrete data assimilation problem: given an external action $q \in (\Xi^N)^K$, determine the initial function $\rho \in \Xi^N$ so that

$$S_\lambda(\rho) = \inf\{S_\lambda(y) \mid y \in \Xi^N\}. \qquad (8)$$

Note that this problem is the approximate finite-dimensional analogue of the optimization problem (5).

For the time-dependent functions we define the projection operator on the grid $P_h$ by the formula $P_h f^k = \frac{1}{\tau}\int_{t_k}^{t_{k+1}} P_N f(t)dt$.

Introduce the norms $\| x \|_{X_h} = \left(\tau \sum_{k=1}^K \| x^k \|_2^2\right)^{1/2}$, $\| q \|_{Z_h} = \left(\tau \sum_{k=0}^{K-1} \| q^k \|_{-2}^2\right)^{1/2}$ $[x]_h = \max_{0\le k\le K} \| x^k \|_1$, $\| x \|_{W_h} = [x]_h + \| x \|_{X_h}$. We will need the following statement.

**Theorem 3**[7]**:** *Let $X$ and $Y$ be Banach spaces, $\mathcal{F}: X \to Y$ be a Frechet-differentiable operator and:*
*1) $\mathcal{F}(0) = 0$; 2) the Lipschitz inequality is valid for its derivative*

$$\|\mathcal{F}'(y_1) - \mathcal{F}'(y_2)\|_{X\to Y} \le L\|y_1 - y_2\|_X \quad \forall y_1, y_2 \in B_r(0)$$

*where $B_r(0) = \{y \in X \mid \| y \|_X \le r\}$, $L = L(r) > 0$ is a constant depending on $r$;*

*3) the operator $\mathcal{F}'(0)$ is closed and has the continuous inverse operator $(\mathcal{F}'(0))^{-1}$ determined on the whole $Y$.*
*Then for any $q \in Y$ such that $\| q \|_Y \le \gamma/(M^2 L)$, where $0 < \gamma < 1$, $M = \|(\mathcal{F}'(0))^{-1}\|_{Y\to X}$, there exists a unique element $x$ being the solution to the equation $\mathcal{F}(x) = q$ and satisfying the estimate $\| x \|_X \le \gamma/(ML)$.*

Denote by $F'(y)$ the derivative of $F$ and consider the equation

$$F'(y)x=(q;\rho) \tag{9}$$

which is a system

$$\frac{\Delta D_1^k}{\tau}+P_N J(x_1^k,\Delta y_1^k+l)+P_N J(x_2^k,\Delta y_2^k)+ \\ +P_N J(y_1^k,\Delta x_1^k)+P_N J(y_2^k,\Delta x_2^k)+\sigma\Delta(x_1^k+x_2^k)-\mu\Delta^2 x_1^k=q_1^k, \tag{10}$$

$$\frac{(\Delta-\alpha)D_2^k}{\tau}+P_N J(x_2^k,\Delta y_1^k+l)+P_N J(x_1^k,\Delta y_2^k)+ \\ +P_N J(y_2^k,\Delta x_1^k)+P_N J(y_1^k,\Delta x_2^k)-\alpha P_N J(y_1^k,x_2^k)- \\ -\alpha P_N J(x_1^k,y_2^k)+\sigma\Delta(x_1^k+x_2^k)-\mu\Delta^2 x_2^k+\mu_1\Delta x_2^k-\sigma_1 x_2^k=q_2^k, \\ x^k\in\Xi^N,\quad k=\overline{0,K},\quad x^0=\rho\in\Xi^N. \tag{11}$$

**Theorem 4:** *If (6) is valid, then the solution of equation (9) satisfies* $\|x\|_{W_h}\le c_2\left(\|\rho\|_1^2+\|q\|_{Z_h}^2\right)^{1/2}$, *where* $c_2>0$ *depends on* $\|y\|_{W_h}$ *only.*

**Proof:** By taking the inner product of (10), (11) by $\tau x^{k+1}$ in $L_2^0\times L_2^0$, we have

$$\frac{1}{2}\|x^{k+1}\|_1^2+\frac{1}{2}\|D^k\|_1^2+\frac{\alpha}{2}\|x_2^{k+1}\|^2+\frac{\alpha}{2}\|D_2^k\|^2+\tau\mu\|x^{k+1}\|_2^2$$
$$=\frac{1}{2}\|x^k\|_1^2+\frac{\alpha}{2}\|x_2^k\|^2-\tau(q^k,x^{k+1})+\tau(J(x_1^k,\Delta y_1^k+l) \\ +J(x_2^k,\Delta y_2^k)+J(y_1^k,\Delta x_1^k)+J(y_2^k,\Delta x_2^k),x_1^{k+1}) \\ +\tau(J(x_2^k,\Delta y_1^k+l)+J(x_1^k,\Delta y_2^k)+J(y_2^k,\Delta x_1^k) \\ +J(y_1^k,\Delta x_2^k)-\alpha J(y_1^k,x_2^k)-\alpha J(x_1^k,y_2^k),x_2^{k+1}) \\ +\tau\mu(\Delta^2 D^k,x^{k+1})+\tau\sigma(\Delta(x_1^k+x_2^k),x_1^{k+1}+x_2^{k+1}) \\ +\tau\mu_1(\Delta x_2^k,x_2^{k+1})-\tau\sigma_1(x_2^k,x_2^{k+1}).$$

Using (6), Lemma 1 and Young inequality, we estimate the following quantities:

$$\tau\mu\,|(\Delta^2 D^k,x^{k+1})|\le\tau\mu\sqrt{\Lambda_N}\,\|D^k\|_1\|x^{k+1}\|_2\le\frac{1}{4}\|D^k\|_1^2+\tau^2\mu^2\Lambda_N\|x^{k+1}\|_2^2 \\ \le\frac{1}{4}\|D^k\|_1^2+\tau(\mu-\nu)\|x^{k+1}\|_2^2,$$

$$\tau\left|\left(J(x_2^k,\Delta y_2^k)+J(y_2^k,\Delta x_2^k),x_1^{k+1}\right)\right|\le c\tau\|x_1^{k+1}\|_2\|\nabla x_2^k\|_{L_4(S)}\|\nabla y_2^k\|_{L_4(S)} \\ \le c\tau\|x_1^{k+1}\|_2\|x_2^k\|_1^{1/2}\left(\|x_2^{k+1}\|_2^{1/2}+\Lambda_N^{1/4}\|D_2^k\|_1^{1/2}\right)\|\nabla y_2^k\|_{L_4(S)} \\ \le\frac{\tau\nu}{6}\|x^{k+1}\|_2^2+\frac{1}{64}\|D_2^k\|_1^2+c\tau\|x_2^k\|_1^2\|\nabla y_2^k\|_{L_4(S)}^4.$$

By applying the similar arguments we obtain the inequality

$$\|x^{k+1}\|_1^2+\alpha\|x_2^{k+1}\|^2+\tau\nu\|x^{k+1}\|_2^2\le \\ \le(1+c\tau(1+\|y^k\|_1^2\|y^k\|_2^2))\|x^k\|_1^2+\alpha\|x_2^k\|_1^2+c\tau\|q^k\|_{-2}^2$$

which implies the estimate $\|x\|_{W_h}^2\le c\exp\left(\|y\|_{W_h}^4\right)\left(\|\rho\|_1^2+\|q\|_{Z_h}^2\right)$.

**Lemma 4:** *For* $F'$ *the Lipschitz inequality is valid* $\|F'(y)-F'(z)\|_{W_h\to Z_h\times V_1}\le L\|y-z\|_{W_h}$ *where* $L$ *is a positive constant not depending on* $\tau$ *and* $N$.

**Proof:** Set $s=y-z$. For every $x\in(\Xi^N)^{K+1}$ we have the equality $F'(y)x-F'(z)x=(\xi;0)$ where

$$\xi_1^k=P_N(J(x_1^k,\Delta s_1^k)+J(x_2^k,\Delta s_2^k)+J(s_1^k,\Delta x_1^k)+J(s_2^k,\Delta x_2^k)), \\ \xi_2^k=P_N(J(x_2^k,\Delta s_1^k)+J(x_1^k,\Delta s_2^k)+J(s_2^k,\Delta x_1^k)+J(s_1^k,\Delta x_2^k) \\ -\alpha J(s_1^k,x_2^k)-\alpha J(x_1^k,s_2^k)).$$

Let $r\in(\Xi^N)^K$. Using the estimates such as

$$\left|\left(J(x_1^k,\Delta s_1^k)+J(s_1^k,\Delta x_1^k),r_1^k\right)\right|\le c\|r_1^k\|_2\|\nabla s_1^k\|_{L_4(S)}\|\nabla x_1^k\|_{L_4(S)} \\ \le c\|r_1^k\|_2\|s_1^k\|_1^{1/2}\|s_1^k\|_2^{1/2}\|x_1^k\|_1^{1/2}\|x_1^k\|_2^{1/2},$$

we get the inequality

$$\tau\left|\sum_{k=0}^{K-1}(\xi^k,r^k)\right|\le L\left(\tau\sum_{k=0}^{K-1}\|r^k\|_2^2\right)^{1/2}[s]_h^{1/2}\|s\|_{X_h}^{1/2}[x]_h^{1/2}\|x\|_{X_h}^{1/2}.$$

Now verify the following assertions of the stability and the convergence of scheme (7).

**Theorem 5:** *If (6) is valid and* $x$ *is the solution to the equation* $F(x)=(q;\rho)$ *and* $y$ *is the solution to the equation* $F(y)=(q+dq;\rho+d\rho)$ *then for any* $\varepsilon>0$ *there exists* $\delta>0$ *depending on* $\varepsilon$ *and* $\|x\|_{W_h}$ *only and such that* $\|x-y\|_{W_h}\le\varepsilon$ *for* $\left(\|d\rho\|_1^2+\|dq\|_{Z_h}^2\right)^{1/2}\le\delta$.

**Proof:** Denote $z=y-x$ and consider the operator $\mathcal{F}(z)=F(x+z)-F(x)$ acting from $W_h$ into $Z_h\times V_1$. By Lemma 4 the derivative $\mathcal{F}'(z)=F'(x+z)$ satisfies the Lipschitz inequality and virtue of Theorem 4 the norm of the inverse operator $(\mathcal{F}'(0))^{-1}$ satisfies the estimate $\|(\mathcal{F}'(0))^{-1}\|_{Z_h\times V_1\to W_h}\le c_2$. Thus, $\mathcal{F}$ satisfies all the conditions of Theorem 3. Since the solution to (7) is unique, then for completing the proof it is sufficient to assume $\delta=\gamma(1-\gamma)/(c_2^2L)$, where $\gamma=\min\{\varepsilon c_2L,1/2\}$.

**Theorem 6:** *Let (6) be valid,* $x_0\in V_2$, $f\in Z$, *a function* $x\in W$ *be the solution to problem (1)-(3),* $w^k=P_Nx(t_k)$, $k=\overline{0,K}$, *a grid function* $y$ *be the solution to the equation* $F(y)=\left(P_hf;P_Nx_0\right)$. *Then we have the convergence* $\|y-w\|_{W_h}\to0$ *for* $\tau\to0$, $N\to\infty$. *If in addition* $u_j^0$, $v_j^0$, $j=1,2$, *belong to* $L_2(G)$, *then* $I(A(y))\to I(x)$ *as* $\tau\to0$, $N\to\infty$.

**Proof:** Applying the operator $P_h$ to both sides of (1) and (2), we obtain the equation $F(w)=(P_h f+dq;P_N x_0)$. Denote $z=P_N x$, $z_t=\frac{\partial z}{\partial t}$, $x_t=\frac{\partial x}{\partial t}$, $[x]=\max_{0\le t\le T}\|x(t)\|_2$ and estimate the typical terms in the residual $dq$:

$$d^k=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}(\Delta^2 z(t)-\Delta^2 w^k)dt=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}(t_{k+1}-t)\Delta^2 z_t dt,$$

$$\|d^k\|_2=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}(t_{k+1}-t)\|z_t\|_2\,dt\le\frac{\sqrt{\Lambda_N}}{\tau}\int_{t_k}^{t_{k+1}}(t_{k+1}-t)\|z_t\|\,dt\le\sqrt{\frac{\tau\Lambda_N}{3}}\left(\int_{t_k}^{t_{k+1}}\|x_t\|_1^2\,dt\right)^{1/2},$$

so $\|d\|_{Z_h}\le c\sqrt{\tau}\|x_t\|_{Y_1}$. Further we estimate

$$\delta^k=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}P_N\left(J(w^k,\Delta w^k)-J(x,\Delta x)\right)dt=\beta^k+\eta^k,$$

$$\beta^k=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}P_N\left(J(z,\Delta z)-J(x,\Delta x)\right)dt=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}P_N\left(J(z-x,\Delta x)+J(z,\Delta z-\Delta x)\right)dt,$$

$$\|\beta^k\|_{-2}\le\frac{c}{\tau}\int_{t_k}^{t_{k+1}}\|x\|_2\|x-z\|_2\,dt\le\frac{c}{\sqrt{\tau}}\left(\int_{t_k}^{t_{k+1}}\|x\|_2^2\|x-z\|_2^2\,dt\right)^{1/2},$$

$$\|\beta\|_{Z_h}\le c\|x-P_N x\|_Y\,[x]\le\frac{c}{\sqrt{\Lambda_{N+1}}}\|x\|_X\,[x].$$

For the second term included in $\delta^k$ we have

$$\eta^k=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}P_N\left(J(w^k,\Delta w^k)-J(z,\Delta z)\right)dt=\frac{1}{\tau}\int_{t_k}^{t_{k+1}}(t-t_{k+1})P_N\frac{\partial J(z,\Delta z)}{\partial t}dt,$$

$$\|\eta^k\|_{-2}\le\frac{c}{\tau}\int_{t_k}^{t_{k+1}}(t_{k+1}-t)\|z\|_2\|z_t\|_2\,dt\le\frac{c\sqrt{\Lambda_N}}{\tau}\int_{t_k}^{t_{k+1}}(t_{k+1}-t)\|x\|_2\|z_t\|_1\,dt$$

$$\le\sqrt{c\tau\Lambda_N}\left(\int_{t_k}^{t_{k+1}}\|x\|_2^2\|x_t\|_1^2\,dt\right)^{1/2},\quad\|\eta\|_{Z_h}\le c\sqrt{\tau}\|x_t\|_{Y_1}\,[x].$$

Applying the similar arguments we see that $\|dq\|_{Z_h}\to 0$ for $\tau\to 0$, $N\to\infty$. It is not difficult to see that $\|w\|_{W_h}\le c\|x\|_W$. By Theorem 5 we have the convergence $\|y-w\|_{W_h}\to 0$ for $\tau\to 0$, $N\to\infty$. Now we estimate $\|A(y)-x\|_{Y_1}\le\|A(y)-A(w)\|_{Y_1}+\|A(w)-z\|_{Y_1}+\|z-x\|_{Y_1}$, where $\|A(y)-A(w)\|_{Y_1}\le c\|y-w\|_{W_h}$,

$$A(w)-z=\frac{t_{k+1}-t}{\tau}(z(t_k)-z(t))+\frac{t-t_k}{\tau}(z(t_{k+1})-z(t))\quad for\ t\in[t_{k-1},t_k],$$

$$\int_{t_k}^{t_{k+1}}\|A(w)-z\|_1^2\,dt\le c\int_{t_k}^{t_{k+1}}\frac{(t_{k+1}-t)^2}{\tau^2}\left\|\int_{t_k}^{t}z_t dt'\right\|_1^2 dt\le c\tau^2\int_{t_k}^{t_{k+1}}\|z_t\|_1^2\,dt,$$

$$\|A(w)-z\|_{Y_1}\le c\tau\sqrt{\tau}\|x_t\|_{Y_1},\quad\|z-x\|_{Y_1}\le\Lambda_{N+1}^{-1}\|x\|_X.$$

Since $A(y)\to x$ strongly in $Y_1$, then $I(A(y))\to I(x)$ as $\tau\to 0$, $N\to\infty$.

## Results and Discussion

The main result of the paper is the following theorem on the convergence of numerical solutions to the data assimilation problem.

**Theorem 7:** *Let the data $u_j^0$, $v_j^0$, $j=1,2$, belong to $L_2(G)$ and the sequence of functions $\rho_n$ is such that:*

*1) $\rho_n$ is the solution to data assimilation problem (8) with $q=P_h f$, $N=N_n$, grid time step $\tau=\tau_n$, and the regularization parameter $\lambda=\lambda_n\ge 0$;*

*2) $\tau_n\to 0$, $N_n\to\infty$, $\lambda_n\to\lambda_0>0$ for $n\to\infty$, and (6) holds.*

*Then $\rho_n$ contains a subsequence converging strongly in $V_2$ to the solution of problem (5) with the same data and $\lambda=\lambda_0$.*

**Proof:** Denote $m_\lambda=\inf\{J_\lambda(y)\,|\,y\in V_2\}$, $s_\lambda=\inf\{S_\lambda(y)\,|\,y\in\Xi^N\}$. We show that for any $\lambda_0\ge 0$ the following inequality holds

$$\limsup_{\tau\to 0,N\to\infty,\lambda\to\lambda_0} s_\lambda\le m_{\lambda_0}. \qquad (12)$$

Indeed, by the definition of the infimum, for any $\varepsilon>0$ there exists a vector function $y\in V_2$ such that $J_{\lambda_0}(y)\le m_{\lambda_0}+\varepsilon/2$. Due to Theorem 6, there exist $\tau^0>0$, $N^0\in\mathbb{N}$, and $d>0$ such that $S_\lambda(P_N y)\le J_{\lambda_0}(y)+\varepsilon/2\le m_{\lambda_0}+\varepsilon$ for all $\tau\le\tau^0$, $N\ge N^0$, and $|\lambda-\lambda_0|\le d$. Then $s_\lambda\le S_\lambda(P_N y)\le m_{\lambda_0}+\varepsilon$, which gives (12). For $\lambda_0>0$ the sequence $\rho_n$ is bounded in $V_2$. Select from it a subsequence $\rho_n\to x_0$ converging weakly in $V_2$ and strongly in $V_1$. Denote

$x=\Phi^{-1}(f;x_0)$, $y_n=F^{-1}(P_h f;P_{N_n}x_0)$, $z_n=F^{-1}(P_h f;\rho_n)$.

By Theorem 6 we have

$$I(A(y_n))\to I(x)\quad\text{for}\quad n\to\infty \qquad (13)$$

and $\|y_n\|_{W_h}\le c\|x\|_W$ for all sufficiently large $n$. Since $\|P_{N_n}x_0-\rho_n\|_1\to 0$, then Theorem 5 implies the convergence $\|y_n-z_n\|_{W_h}\to 0$ as $n\to\infty$. Consequently, $A(y_n)-A(z_n)\to 0$ strongly in $L_2(0,T;V_1)$, then $I(A(y_n))-I(A(z_n))\to 0$. Taking into account (13), we see that

$$S_0(\rho_n)\to J_0(x_0)\quad\text{for}\quad n\to\infty. \qquad (14)$$

A weak convergence of $\rho_n$ to $x_0$ in $V_2$ implies that $\liminf_{n\to\infty}\|\rho_n-x_0^a\|_2\ge\|x_0-x_0^a\|_2$. Taking into account (14) and the convergence of $\lambda_n\to\lambda_0$, we get

$$\liminf_{n\to\infty} S_{\lambda_n}(\rho_n) \geq J_{\lambda_0}(x_0). \qquad (15)$$

But $\rho_n$ are the solutions of (8), so $S_{\lambda_n}(\rho_n) = s_{\lambda_n}$. From (12) we have $\limsup_{n\to\infty} S_{\lambda_n}(\rho_n) \leq m_{\lambda_0} \leq J_{\lambda_0}(x_0)$. Comparing (15) with the last inequality we see that

$$\lim_{n\to\infty} S_{\lambda_n}(\rho_n) = J_{\lambda_0}(x_0) = m_{\lambda_0}, \qquad (16)$$

that is $x_0$ is the solution of data assimilation problem (5). In addition, from (14) and (16) we find that

$\lim_{n\to\infty} \lambda_n \| \rho_n - x_0^a \|_2^2 = \lambda_0 \| x_0 - x_0^a \|_2^2$, then

$\lim_{n\to\infty} \| \rho_n - x_0^a \|_2^2 = \| x_0 - x_0^a \|_2^2$, so $\rho_n \to x_0$ strongly in $V_2$.

Notice that the arguments of Theorem 7 imply that if $\lambda_0 = 0$ and the sequence $\rho_n$ is bounded in $V_2$, then $\rho_n$ contains a subsequence weakly converging in $V_2$ to the solution to problem (5) with the same data and $\lambda = 0$.

## Conclusion

In this paper we have considered a method of approximate solution of the data assimilation problem for the two-layer quasigeostrophic atmospheric general circulation model and have proved the convergence of numerical solutions to the exact solutions of the optimization problem. One can hope that in future data assimilation techniques will find application in various branches of science [8–10].

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